SUPPLEMENTARY MATERIAL

CHAPTER 7

7.6.3 $\int (px+q)\sqrt{ax^2+bx+c}\;dx$.

We choose constants A and B such that

$$px + q = A\left[\frac{d}{dx}(ax^2+bx+c)\right] + B$$

$$= A(2ax + b) + B$$

Comparing the coefficients of x and the constant terms on both sides, we get

$$2aA = p \text{ and } Ab + B = q$$

Solving these equations, the values of A and B are obtained. Thus, the integral reduces to

$$A\int (2ax+b)\sqrt{ax^2+bx+c}\;dx + B\int\sqrt{ax^2+bx+c}\;dx$$

$$= AI_1 + BI_2$$

where $\quad I_1 = \int (2ax+b)\sqrt{ax^2+bx+c}\;dx$

Put $ax^2 + bx + c = t$, then $(2ax + b)dx = dt$

So $\quad I_1 = \frac{2}{3}(ax^2+bx+c)^{\frac{3}{2}} + C_1$

Similarly, $\quad I_2 = \int\sqrt{ax^2+bx+c}\;dx$

is found, using the integral formulae discussed in [7.6.2, Page 328 of the textbook].

Thus $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ is finally worked out.

Example 25 Find $\int x\sqrt{1+x-x^2}\,dx$

Solution Following the procedure as indicated above, we write

$$x = A\left[\frac{d}{dx}\left(1+x-x^2\right)\right]+B$$

$$= A\,(1-2x)+B$$

Equating the coefficients of x and constant terms on both sides,

We get $-2A = 1$ and $A + B = 0$

Solving these equations, we get $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Thus the integral reduces to

$$\int x\sqrt{1+x-x^2}\,dx = -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}I_1 + \frac{1}{2}I_2 \qquad (1)$$

Consider $\qquad I_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx$

Put $1+x-x^2 = t$, then $(1-2x)dx = dt$

Thus $\quad I_1 = \int (1-2x)\sqrt{1+x-x^2}\,dx = \int t^{\frac{1}{2}}dt = \frac{2}{3}t^{\frac{3}{2}} + C_1$

$$= \frac{2}{3}\left(1+x-x^2\right)^{\frac{3}{2}} + C_1, \text{ where } C_1 \text{ is some constant.}$$

Further, consider $\quad I_2 = \int \sqrt{1+x-x^2}\,dx = \int \sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}\,dx$

Put $x-\frac{1}{2}=t.$ Then $dx = dt$

Therefore, $\quad I_2 = \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - t^2}\,dt$

$$= \frac{1}{2}t\sqrt{\frac{5}{4}-t^2} + \frac{1}{2}\cdot\frac{5}{4}\sin^{-1}\frac{2t}{\sqrt{5}} + C_2$$

$$= \frac{1}{2}\frac{(2x-1)}{2}\sqrt{\frac{5}{4}-(x-\frac{1}{2})^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2$$

$$= \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C_2,$$

where C_2 is some constant.

Putting values of I_1 and I_2 in (1), we get

$$\int x\sqrt{1+x-x^2}\,dx = -\frac{1}{3}(1+x-x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2}$$

$$+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + C,$$

where $\quad C = -\frac{C_1+C_2}{2}$ is another arbitrary constant.

Insert the following exercises at the end of EXERCISE 7.7 as follows:

12. $x\sqrt{x+x^2}$ 13. $(x+1)\sqrt{2x^2+3}$ 14. $(x+3)\sqrt{3-4x-x^2}$

Answers

12. $\frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{(2x+1)\sqrt{x^2+x}}{8} + \frac{1}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + C$

13. $\frac{1}{6}(2x^2+3)^{\frac{3}{2}} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3\sqrt{2}}{4}\log\left|x+\sqrt{x^2+\frac{3}{2}}\right| + C$

14. $-\frac{1}{3}(3-4x-x^2)^{\frac{3}{2}} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + \frac{(x+2)\sqrt{3-4x-x^2}}{2} + C$

CHAPTER 10

10.7 Scalar Triple Product

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors. The scalar product of $\vec{a}$ and $(\vec{b}\times\vec{c})$, i.e., $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ in this order and is denoted by $[\vec{a}, \vec{b}, \vec{c}]$ (or $[\vec{a}\,\vec{b}\,\vec{c}]$). We thus have

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$$

Observations

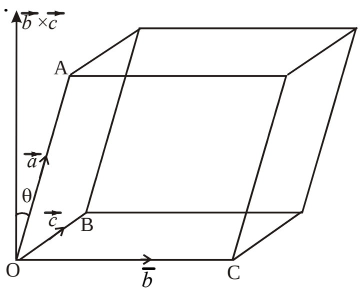

Fig.10.28

1. Since $(\vec{b}\times\vec{c})$ is a vector, $\vec{a}\cdot(\vec{b}\times\vec{c})$ is a scalar quantity, i.e. $[\vec{a}, \vec{b}, \vec{c}]$ is a scalar quantity.
2. Geometrically, the magnitude of the scalar triple product is the volume of a parallelopiped formed by adjacent sides given by the three

vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ (Fig. 10.28). Indeed, the area of the parallelogram forming the base of the parallelopiped is $\left|\vec{b}\times\vec{c}\right|$. The height is the projection of $\vec{a}$ along the normal to the plane containing $\vec{b}$ and $\vec{c}$ which is the magnitude of the component of $\vec{a}$ in the direction of $\vec{b}\times\vec{c}$ i.e., $\dfrac{\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|}{\left|(\vec{b}\times\vec{c})\right|}$. So the required volume of the parallelopiped is $\dfrac{\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|}{\left|(\vec{b}\times\vec{c})\right|}\,|\vec{b}\times\vec{c}| = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$,

3. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\,\hat{i} + (b_3c_1 - b_1c_3)\,\hat{j} + (b_1c_2 - b_2c_1)\,\hat{k}$$

and so

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

4. If $\vec{a}, \vec{b}$ and $\vec{c}$ be any three vectors, then

$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$

(cyclic permutation of three vectors does not change the value of the scalar triple product).

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then, just by observation above, we have

$$[\vec{a},\vec{b},\vec{c}\,]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\,(b_2c_3 - b_3c_2) + a_2\,(b_3c_1 - b_1c_3) + a_3\,(b_1c_2 - b_2c_1)$$

$$= b_1\,(a_3c_2 - a_2c_3) + b_2\,(a_1c_3 - a_3c_1) + b_3\,(a_2c_1 - a_1c_2)$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= [\vec{b},\vec{c},\vec{a}\,]$$

Similarly, the reader may verify that

$$= [\vec{a},\vec{b},\vec{c}\,] = [\vec{c},\vec{a},\vec{b}\,]$$

Hence $\qquad [\vec{a},\vec{b},\vec{c}\,] = [\vec{b},\vec{c},\vec{a}\,] = [\vec{c},\vec{a},\vec{b}\,]$

5. In scalar triple product $\vec{a}.(\vec{b}\times\vec{c})$, the dot and cross can be interchanged. Indeed,

$$\vec{a}.(\vec{b}\times\vec{c}) = [\vec{a},\vec{b},\vec{c}\,] = [\vec{b},\vec{c},\vec{a}\,] = [\vec{c},\vec{a},\vec{b}\,] = \vec{c}.(\vec{a}\times\vec{b}) = (\vec{a}\times\vec{b}).\vec{c}$$

6. $= [\vec{a},\vec{b},\vec{c}\,] = -\,[\vec{a},\,\vec{c},\vec{b}\,]$. Indeed

$$= [\vec{a},\vec{b},\vec{c}\,] = \vec{a}.(\vec{b}\times\vec{c})$$

$$= \vec{a}.(-\,\vec{c}\times\vec{b})$$

$$= -\,(\vec{a}.(\vec{c}\times\vec{b}))$$

$$= -\left[\vec{a},\vec{c},\vec{b}\right]$$

7. $[\vec{a}, \vec{a}, \vec{b}] = 0$. Indeed

$$[\vec{a}, \vec{a}, \vec{b}] = [\vec{a}, \vec{b}, \vec{a},]$$

$$= [\vec{b}, \vec{a}, \vec{a}]$$

$$= \vec{b} \,.\, (\vec{a} \times \vec{a})$$

$$= \vec{b} \,.\, \vec{0} = 0. \qquad \text{(as } \vec{a} \times \vec{a} = \vec{0}\text{)}$$

Note: The result in 7 above is true irrespective of the position of two equal vectors.

10.7.1 Coplanarity of Three Vectors

Theorem 1 Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Proof Suppose first that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

If $\vec{b}$ and $\vec{c}$ are parallel vectors, then, $\vec{b} \times \vec{c} = \vec{0}$ and so $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

If $\vec{b}$ and $\vec{c}$ are not parallel then, since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, $\vec{b} \times \vec{c}$ is perpendicular to $\vec{a}$.

So $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

Conversely, suppose that $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. If $\vec{a}$ and $\vec{b} \times \vec{c}$ are both non-zero, then we conclude that $\vec{a}$ and $\vec{b} \times \vec{c}$ are perpendicular vectors. But $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$. Therefore, $\vec{a}$ and $\vec{b}$ and $\vec{c}$ must lie in the plane, i.e. they are coplanar. If $\vec{a} = 0$, then $\vec{a}$ is coplanar with any two vectors, in particular with $\vec{b}$ and $\vec{c}$. If $(\vec{b} \times \vec{c}) = 0$, then $\vec{b}$ and $\vec{c}$ are parallel vectors and so, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar since any two vectors always lie in a plane determined by them and a vector which is parallel to any one of it also lies in that plane.

Note: Coplanarity of four points can be discussed using coplanarity of three vectors. Indeed, the four points A, B, C and D are coplanar if the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Example 26 Find $\vec{a}.(\vec{b}\times\vec{c})$, *if* $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$.

Solution We have $\vec{a}.(\vec{b}\times\vec{c})=\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}=-10.$

Example 27 Show that the vectors

$\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=-2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}+5\hat{k}$ are coplanar.

Solution We have $\vec{a}.(\vec{b}\times\vec{c})=\begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}=0.$

Hence, in view of Theorem 1, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors.

Example 28 Find λ if the vectors

$\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{c}=\lambda\hat{i}+7\hat{j}+3\hat{k}$ are coplanar.

Solution Since $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar vectors, we have $\left[\vec{a}, \vec{b}, \vec{c}\right]=0$, i.e.,

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ \lambda & 7 & 3 \end{vmatrix}=0.$$

$\Rightarrow \quad 1(-3+7)-3(6+\lambda)+1(14+\lambda)=0$

$\Rightarrow \quad \lambda=0.$

Example 29 Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-(\hat{j}+\hat{k})$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$, respectively are coplanar.

Solution We know that the four points A, B, C and D are coplanar if the three vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar, i.e., if

$$\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right]=0$$

Now $\overrightarrow{AB} = -(\hat{j}+\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k}) = -4\hat{i}-6\hat{j}-2\hat{k}$

$\overrightarrow{AC} = (3\hat{i}+9\hat{j}+4\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k}) = -\hat{i}+4\hat{j}+3\hat{k}$

and $\overrightarrow{AD} = 4(-\hat{i}+\hat{j}+\hat{k}) - (4\hat{i}+5\hat{j}+\hat{k}) = -8\hat{i}-\hat{j}+3\hat{k}$

Thus $\left[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0.$

Hence A, B, C and D are coplanar.

Example 30 Prove that $\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$.

Solution We have

$$\left[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\right] = (\vec{a}+\vec{b}).((\vec{b}+\vec{c})\times(\vec{c}+\vec{a}))$$

$$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$$

$$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}) \qquad (\text{as } \vec{c}\times\vec{c}=\vec{0})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{a}\right] + \left[\vec{a}, \vec{c}, \vec{a}\right] + \left[\vec{b}, \vec{b}, \vec{c}\right] + \left[\vec{b}, \vec{b}, \vec{a}\right] + \left[\vec{b}, \vec{c}, \vec{a}\right]$$

$$= 2\left[\vec{a}, \vec{b}, \vec{c}\right] \qquad \text{(Why?)}$$

Example 31 Prove that $\left[\vec{a}, \vec{b}, \vec{c}+\vec{d}\right] = \left[\vec{a}, \vec{b}, \vec{c}\right] + [\vec{a}, \vec{b}, \vec{d}]$

Solution We have

$$\left[\vec{a}, \vec{b}, \vec{c}+\vec{d}\right] = \vec{a}.(\vec{b}\times(\vec{c}+\vec{d}))$$

$$= \vec{a}.(\vec{b}\times\vec{c}+\vec{b}\times\vec{d})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{d})$$

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{d}\right].$$

Exercise 10.5

1. Find $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ if $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $c = 3i + j - 2\hat{k}$

   (Ans. 24)

2. Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar.

3. Find λ if the vectors $\hat{i} - \hat{j} + \hat{k}$, $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + \lambda\hat{j} - 3\hat{k}$ are coplanar.

   (Ans. $\lambda = 15$)

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ Then

   (a) If $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar (Ans. $c_3 = 2$)

   (b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

5. Show that the four points with position vectors $4\hat{i} + 8\hat{j} + 12\hat{k}$, $2\hat{i} + 4\hat{j} + 6\hat{k}$, $3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

6. Find x such that the four points A (3, 2, 1) B (4, x, 5), C (4, 2, -2) and D (6, 5, -1) are coplanar. (Ans. $x = 5$)

7. Show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar if $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are coplanar.